THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2023-24
Lecture 3 practice problems solution
27th September 2023

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **zdmu@math.cuhk.edu.hk** if you have any questions.

1. (a) Recall that a bijective function $T : S \to S$ is a function that is both injective and surjective, i.e., for all $y \in S$, there is a unique $x \in S$ so that $T(x) = y$. We know $\mathrm{id}_S \in G$ because the identity function is bijective.

   By definition of bijective functions, it is invertible: one can define $T^{-1} : S \to S$ by declaring $T^{-1}(y)$ to be the unique preimage of y under f. One can see that T^{-1} is again bijective because for any $x \in S$, $T(x) = y$ is the unique value so that $T^{-1}(y) = x$.

   Finally, given bijective functions T, U. Their composition $T \circ U$ is again bijective because for any $z \in S$ there is unique y so that $T(y) = z$. And for this y, there is a unique x so that $U(x) = y$. Therefore x is the unique element so that $T(U(x)) = T(y) = z$.

   (b) Consider a figure $A \in D_n$, i.e. A is a subset of S consisting of n elements. Given any bijective function $T \in G$, since it is one-to-one, $T(A)$ also consists of n elements and so $T(A) \in D_n$. So D_n are invariant.

   (c) Since S is a finite set by assumption, the power set, i.e. the set of all subsets of S can be decomposed into union of D_n for different value of n. Let $A \in D$, if A is the empty set, then it has no element, so $T(A)$ is again the empty set and $f(T(A)) = 0$. Otherwise $A \in D_n$ for some n, i.e. $f(A) = n$, by part (b) we know $T(A)$ also has n element so $f(T(A)) = n$. Hence f is an invariant function.

2. (a) To prove that D is an invariant set, it suffices to prove that for any transformation T (in the respective geometries), $T(\ell)$ is again a straight line. For example, a typical straight line ℓ can be expressed as $\{(x, y) \in \mathbb{R}^2 |\, ax + by + c = 0\}$ for some fixed a, b, c. Meanwhile, a translation T can be written as $T(x, y) = (x + h, y + k) = (x', y')$. Then it is clear that $T(\ell) = \{(x', y') \in \mathbb{R}^2 |\, a(x' - h) + b(y' - k) + c = 0\} = \{ax' + by' + (c - ah - bk) = 0\}$ is again a straight line, so $T(\ell) \in D$.

   As for the Euclidean geometry, recall that any Euclidean transformation can be expressed as a rotation followed by a translation. Since we have shown that D is translational invariant, it suffices to prove that D is also rotational invariant. The way to express a rotation in $\mathbb{R}^2$ is essentially by a matrix as in question 3. But equivalently, we can work with $\mathbb{C}$ instead and use multiplication $T(z) = e^{i\theta} z$. We will use a different way to represent a straight line: $\ell = \{z |\, z = z_0 + tz_1, t \in \mathbb{R}\}$, where z_0, z_1 are fixed number in $\mathbb{C}$. And therefore we can write $T(\ell) = \{z' |\, z' = e^{i\theta} z = e^{i\theta} z_0 + te^{i\theta} z_1, t \in \mathbb{R}\}$ which is again a straight line.

(b) In the translational geometry, the slope of a straight line $\ell = \{(x,y) \in \mathbb{R}^2 | \, ax + by + c = 0\}$ can be expressed as $f(\ell) = -\frac{a}{b}$ if $b \neq 0$ and $f(\ell) = \infty$ if $b = 0$. Under a translation T, we see that $T(\ell)$ has slope also given by $-\frac{a}{b}$ by part (a). Therefore $f(T(\ell)) = f(\ell)$ and f is invariant.

(c) In the rotational geometry, let's consider the rotation counterclockwise by $90°$, given by $T(z) = iz$. In the expression $\ell = \{z | \, z = z_0 + tz_1\}$, the slope of ℓ is given the "slope" of the number $z_1 = x + yi$, here $f(\ell) = \frac{y}{x}$ if $x \neq 0$ and $f(\ell) = \infty$ if $x = 0$. Then the slope of $T(\ell)$ in this case would be the "slope" of $iz_1 = -y + xi$, which is $f(T(\ell)) = -\frac{x}{y}$, which is clearly not the same as $f(\ell)$ for most values of x, y. Therefore f is not invariant in rotational geometry.

3. The solution to Q3 will be rather sketchy, because there is a lot of details that make use of difficult results from topology. It is recommended that you draw some pictures to understand the geometry. This question might be too difficult for our syllabus, you can skip this in revision.

(a) Taking $\theta = 0$ in $SO(2, \mathbb{R})$ gives the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which sends any vector v to $Iv = v$, so it is the identity map in this geometry. Notice that if we use the direction vector v to represent a line ℓ, then any nonzero multiple kv will represent the same line.

Any transformation is invertible, this is because the matrix M has determinant $\cos^2\theta + \sin^2\theta = 1$. The inverse matrix M^{-1} is given by

$$M^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} \in SO(2, \mathbb{R}).$$

Conceptually, the matrix M is just rotation by θ, therefore its inverse is given by rotation by $-\theta$.

Finally, one can check by sum of angle formula that

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \cos(\theta+\varphi) & -\sin(\theta+\varphi) \\ \sin(\theta+\varphi) & \cos(\theta+\varphi) \end{pmatrix} \in SO(2, \mathbb{R})$$

Again, this just means that rotation by φ followed by rotation by θ is the same as rotation by $\theta + \varphi$.

This shows that (S, G) is a geometry. Technically we have to work with $PSO(2, \mathbb{R})$, but without getting too abstract, you can convince yourself that all arguments above work after taking quotient by $\{\pm I\}$.

(b) To show that two geometries are isomorphic, we need to find a bijection between

$$S^1 \Leftrightarrow \{\text{all lines through origin in } \mathbb{R}^2\} =: \mathbb{RP}^1$$

Again any lines can be represented by a vector $v \in \mathbb{R}^2$, since the length of the vectors do not matter and we only care about its direction, we can restrict our attention to those vectors with unit lengths. From this, we immediately have a map $p : S^1 \to \mathbb{RP}^1$ by sending v to the line ℓ generated by v. However, this is not quite a bijection, because v and $-v$ will give the same line. This is only a 2-to-1 map. Now imagine $\mathbb{RP}^1$ as being "half" of a circle, but with both ends connected, this is still a circle geometrically. And we have the following diagram,

$$\begin{tikzcd} S^1 \arrow[r, "z \mapsto z^2"] \arrow[dr, "p"'] & S^1 \arrow[d, "\mu"] \\ & \mathbb{RP}^1 \end{tikzcd}$$

Here μ is the bijection between the two geometries, p is the map sending a vector to the corresponding line ℓ. The map p is essentially the same as $z \mapsto z^2$ or $e^{i\theta} \mapsto e^{2i\theta}$, where a circle winds around itself twice (which is the same as doubling the angle θ). Now recall that $SO(2, \mathbb{R})$ is really the set of all rotations on $\mathbb{R}^2$, but this is nothing by S^1 itself. The geometry (S^1, S^1) has transformation given by S^1 rotating itself. Under the map $z \mapsto z^2$, a rotation by θ becomes a rotation by 2θ. So in particular $-I$ = rotation by 180° acts on S^1 in the image by identity. Therefore the transformation group becomes $PSO(2, \mathbb{R}) = SO(2, \mathbb{R})/\{\pm I\}$, this gives an identification between the transformation groups of two geometries.